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PROPAGATION IN A TROPOSPHERIC DUCT WITH A SINGLE-STEP DISCONTINUITY IN THE REFRACTIVE INDEX IN THE DIRECTION OF PROPAGATION

R. H. Ott
Institute for Telecommunication Sciences
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Boulder, Colorado 80303

June 1980

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Final Report for Period November 1979 - May 1980

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A Green's function approach is used to derive an expression for the field inside a laterally inhomogeneous duct. The laterally inhomgeneous duct is assumed to have a single step, and convergence criteria for the step size and number of modes are discussed.

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FOREWORD

This technical report describes the work completed by the Institute for Telecommunication Sciences, U.S. Department of Commerce, Boulder, Colorado, under Project ILIR9209, "Over-the-Horizon Target Detection Feasibility Study." This work has been supported by the Avionics Laboratory Director's Fund.

The work described herein is for the period November 1979 to May 1980, under the direction of Mr. Raymond P. Wasky (AFWAL/AARI-3), Electro-Optics and Reconnaissance Branch, Reconnaissance and Weapon Delivery Division, Air Force Avionics Laboratory, Wright-Patterson Air Force Base, Ohio.

This report was submitted by the author in June 1980.

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SUMMARY

The purpose of this study was to generate a model for use in determining the feasibility of detecting radar signals beyond the normal radar horizon. The mechanisms to be considered were tropospheric ducting and earth diffraction. Until recently, models of tropospheric ducts assumed that the ducts were horizontally homogeneous, which led to significant errors when compared with experimental results. Under this study, ducts are treated as laterally nonuniform stratifications in the lower atmosphere.

A Green's function approach is used to derive an expression for the field inside a laterally inhomogeneous duct. The laterally inhomogeneous duct is assumed to have a single step discontinuity. The formulation for the field, convergence criteria for the step size and the number of modes needed for a solution are discussed.

SECTION I

INTRODUCTION

A solution to a very "sticky" problem is given, using the Green's function method. The extension of the solution for propagation in a uniform medium to a medium composed of steps, to represent the slow variation in refractive index with distance along the direction of propagation, allows a solution for propagation in a laterally inhomogeneous medium. The coupling between normal modes in each region is easily separated out of the solution in the present formulation. The theory could be used to study the problem of propagation of underwater sound waves in shallow water with slowly varying depth.[1] The problem of propagation in a laterally inhomogeneous duct was investigated by Bahar[2] using an iterative solution to Maxwell's equations directly.

SECTION II

ANALYSIS

The geometry of the propagation problem is shown in Figure 1. The term duct refers to the concept of the trapping of modes and the resulting propagation over long distances.

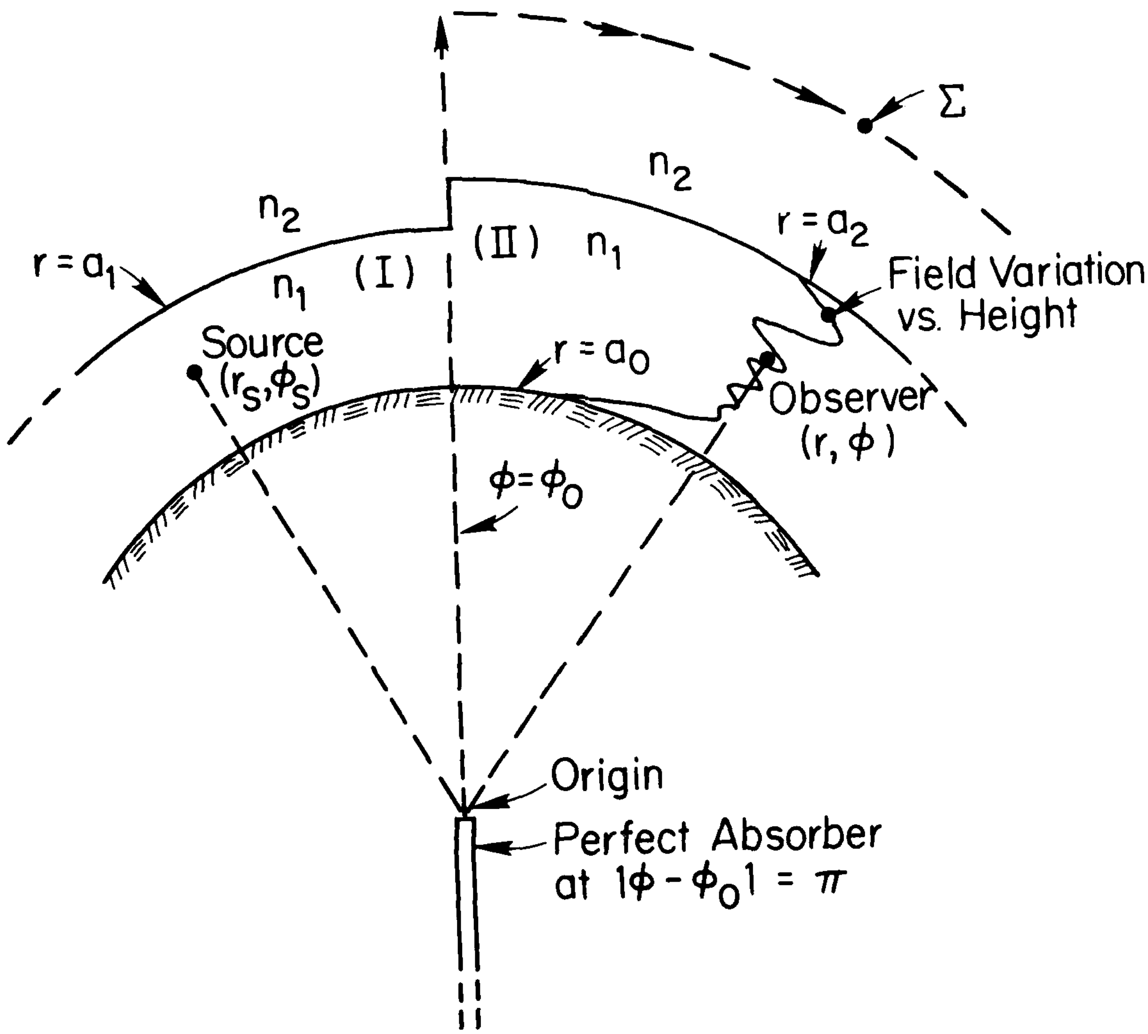

Figure 1. Geometry for single step discontinuity in a tropospheric duct.

Our approach will be to find the Green's function for the bounded waveguide cross section in Figure 1. In particular, we will analyze the effect of the step size, (a_2-a_1), and the refractive index contrast, n_2-n_1.

The electric and magnetic fields will satisfy the two-dimensional, time-harmonic, Helmholtz equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 G}{\partial \phi^2} + k^2 G = \frac{\delta(r-r_s)\delta(\phi-\phi_s)}{r} \quad , \tag{1}$$

$$a_0 \leq r < \infty$$

$$0 \leq \phi \leq 2\pi$$

where (r_s,ϕ_s) and (r,ϕ) are the source and observation coordinates respectively, k is the wave number, and the time dependence is e^{-iwt}. Considering regions (I) and (II) separately, G must meet the periodicity requirement, and we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 G}{\partial \phi^2} + k^2 G = \frac{\delta(r-r_s)}{r}\sum_{n=-\infty}^{\infty}\delta(\phi-\phi_s-2n\pi) \tag{2}$$

We look for solutions to (2) with a singularity at r_s,ϕ_s on each "Riemann sheet", n as

$$G(\underline{r}, \underline{r}') = \sum_{-\infty}^{\infty} G_\infty(\underline{r}, \underline{r}_n'), \quad \underline{r}_n' = (r', \phi_s + 2n\pi)$$

where G_∞ satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G_\infty}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 G_\infty}{\partial \phi^2} + k^2 G_\infty = \frac{\delta(r-r_s)\delta(\phi-\phi_s)}{r} \quad , \tag{3}$$

$$a_0 \leq r < \infty$$

$$-\infty < \phi < \infty \quad .$$

The completeness relation is

$$G_\infty(\underline{r}, \underline{r}_n') = \frac{-1}{2\pi i}\oint g_r(r,r_s;\lambda)g_\phi(\phi, \phi_s;\lambda)d\lambda \tag{4}$$

where the contour (counterclockwise) in (4) is selected to enclose all the singularities in the complex λ-plane.

If we define

$$\nu = \sqrt{\lambda}\ , \qquad \mathrm{Im}(\nu)>0$$
$$d\nu = \frac{1}{2\sqrt{\lambda}}\, d\lambda \tag{5}$$

then the Green's function g_ϕ on an "infinite" angular transmission line is

$$g_\phi(\phi,\ \phi_s;\nu^2) = \frac{e^{i\nu(\phi-\phi_s-2n\pi)}}{2i\nu} \tag{6}$$

with $|g_\phi| \to o$ as $|\nu| \to \infty$. Substituting (6) in (4) gives

$$G_\infty(\underline{r},\ \underline{r}_n') = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_r(r,r_s;\nu^2)e^{i\nu(\phi-\phi_s-2n\pi)} d\nu \tag{7}$$

with Fourier inversion

$$g_r(r,\ r_s;\nu^2) = \int_{-\infty}^{\infty} G_\infty(r,r';\phi,\phi_s+2n\pi)e^{-i\nu(\phi-\phi_s-2n\pi)}\ d\phi. \tag{8}$$

From (3), g_r satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g_r}{\partial r}\right) + k^2 g_r - \frac{\nu^2}{r^2} g_r = \frac{\delta(r-r_s)}{r}\ . \tag{9}$$

With a perfect absorber at $|\phi-\phi_s|=\pi$, Figure 1 corresponds to the case n = o for (7) and (8). Much of the discussion up to this point can be found in the literature.[3]

The field, E(r), in region (II) due to the aperture field in the plane $\phi=\phi_o$ in Figure (1) is obtained from Green's Theorem after integrating over the cylinder Σ at infinity and over the aperture plane $\phi=\phi_o$ yielding

$$E(r) = \int_{a_o}^{\infty} \frac{dr'}{r'}\left[E(r')\frac{\partial G_\infty^{(2)}}{\partial\phi'} - G_\infty^{(2)}\frac{\partial E}{\partial\phi'}\right] \quad , \tag{10}$$

where by analogy with the problem to the left of the aperture

$$G_{\infty}^{(2)}\ (r,r';\phi,\phi') = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_r^{(2)}(r,r';\nu^2)e^{i\nu(\phi-\phi')}\ d\nu, \tag{11}$$

from which it follows that

$$\frac{\partial G_{\infty}^{(2)}}{\partial \phi'} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} g_r^{(2)}(r,r';\nu^2)e^{i\nu(\phi-\phi')}\ \nu\ d\nu\ . \tag{12}$$

We now make the parabolic wave equation assumption

$$\frac{\partial E}{\partial \phi'} \sim ika_1\ E(r') \tag{13}$$

Subsituting (12) and (13) into (10) gives

$$E(r) = \int_{a_o}^{\infty} \frac{dr'}{r'} E(r') \left[\frac{1}{2\pi i} \int_{-\infty}^{\infty} g_r^{(2)}(r,r';\nu^2)e^{i\nu(\phi-\phi')}\nu\ d\nu, \right. \tag{14}$$
$$\left. + \frac{ka_1}{2\pi i} \int_{-\infty}^{\infty} g_r^{(2)}(r,r';\nu^2)e^{i\nu(\phi-\phi')}d\nu \right]$$

and because we are interested in solutions when $\nu \cong ka_1$, (14) becomes

$$E(r) = \frac{ka_1}{\pi i} \int_{a_o}^{\infty} \frac{dr'}{r'} E(r') \int_{-\infty}^{\infty} g_r^{(2)}(r,r';\nu^2)e^{i\nu(\phi-\phi')}d\nu\ . \tag{15}$$

Now, $E(r')$ is the field in the plane $\phi=\phi_o$ due to the point source at r_s, ϕ_s neglecting reflected fields and is given by

$$E(r') = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_r^{(1)}(r',r_s;\nu^2)e^{i\nu(\phi_o-\phi_s)}\ d\nu\ . \tag{16}$$

Substituting (16) into (15) gives the solution

$$E(r) = \frac{ka_1}{2\pi^2} \int_{a_0}^{\infty} \frac{dr'}{r'} \int_{-\infty}^{\infty} d\nu_1 \int_{-\infty}^{\infty} d\nu_2 \; g_r^{(1)}(r', r_s; \nu_1^2) g_r^{(2)}(r, r'; \nu_2^2) e^{i\nu_1(\phi_0 - \phi_s)} \cdot e^{i\nu_2(\phi - \phi_0)} \tag{17}$$

which is easily generalized to 2-step discontinuities as

$$E(r) = \frac{ka_1}{2\pi^2} \int_{a_0}^{\infty} \frac{dr''}{r''} \int_{a_0}^{\infty} \frac{dr'}{r'} \int_{-\infty}^{\infty} d\nu_1 \int_{-\infty}^{\infty} d\nu_2 \int_{-\infty}^{\infty} d\nu_3 \; \cdot g_r^{(1)}(r', r_s; \nu_1^2) \; g_r^{(2)}(r', r'', \nu_2^2) \; g_r^{(3)}(r'', r; \nu_3^2) \; e^{i\nu_1(\phi_1 - \phi_s)} \; e^{i\nu_2(\phi_2 - \phi_1)} \; e^{i\nu_3(\phi_2 - \phi_0)} \tag{18}$$

The Green's function for the source r_s in the duct is given by the "broken" function

$$g(r, r_s; \nu^2) = \begin{cases} T(\nu)[H_\nu^{(1)}(kr_s) + R_1(\nu) H_\nu^{(2)}(kr_s)] \; H_\nu^{(1)}(\hat{k}r), & a_1 \le r < \infty \\ \alpha(\nu)[H_\nu^{(1)}(kr) + R_2(\nu) H_\nu^{(2)}(kr)] \cdot \\ \quad \cdot [H_\nu^{(1)}(kr_s) + R_1(\nu) \; H_\nu^{(2)}(kr_s)], & r_s \le r \le a_1 \\ \alpha(\nu)[H_\nu^{(1)}(kr_s) + R_s(\nu) \; H_\nu^{(2)}(kr_s)] \cdot \\ \quad \cdot [H_\nu^{(1)}(kr) + R_1(\nu) \; H_\nu^{(2)}(kr)], & a_0 \le r \le r_s \end{cases} \tag{19}$$

In (19) $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are Hankel functions of the first and second kind.

where a_1 is the height where the refractive index changes from k to $\hat{k}$ where

$$\hat{k} = k(1 - \Delta n) \tag{20}$$

and a_0 is the radius of the earth and Δn is the refractive index contrast. The Green's function if the source is outside the duct is

$$\hat{g}(r,r_s;\nu^2) = \begin{cases} \hat{\alpha}(\nu)[H_\nu^{(1)}(\hat{k}r_s) + \hat{R}_2(\nu)H_\nu^{(2)}(\hat{k}r_s)]H_\nu^{(1)}(\hat{k}r), & r_s \leq r < \infty \\ \hat{\alpha}(\nu)[H_\nu^{(1)}(\hat{k}r) + \hat{R}_2(\nu)\, H_\nu^{(2)}(\hat{k}r)]H_\nu^{(1)}(\hat{k}r_s), & a_2 \leq r \leq r_s \\ \hat{T}(\nu)[H_\nu^{(1)}(kr) + R_1(\nu)\, H_\nu^{(2)}(kr)]H_\nu^{(1)}(\hat{k}r_s), & a_0 \leq r \leq a_2 \end{cases} \tag{21}$$

where the height of the duct is now labeled a_2, because, as we see in Figure 2, the integration point for Region II eventually lies outside the duct.

From the jump condition

$$\left.\frac{\partial g}{\partial r}\right|_{r=r_s+\varepsilon} - \left.\frac{\partial g}{\partial r}\right|_{r=r_s-\varepsilon} = \frac{-1}{2\pi r_s} \tag{22}$$

and the use of the following Wronskian

$$H_\nu^{(1)'}(kr_s)H_\nu^{(2)}(kr_s) - H_\nu^{(1)}(kr_s)H_\nu^{(2)'}(kr_s) = \frac{4i}{\pi k r_s} \quad , \tag{23}$$

we obtain

$$\alpha(\nu) = \frac{-i}{8[R_2(\nu)-R_1(\nu)]} \tag{24}$$

and the "resonance" condition; i.e., $R_2(\nu) = R_1(\nu)$. From the boundary conditions at $r=a_1$, i.e.,

$$g(r,r_s;\nu^2)\Big|_{r=a_1-\varepsilon} = g(r,r_s;\nu^2)\Big|_{r=a_1+\varepsilon} \tag{25}$$

and

$$\left.\frac{\partial g}{\partial r}\right|_{r=a_1-\varepsilon} = \left.\frac{\partial g}{\partial r}\right|_{r=a_1+\varepsilon} \tag{26}$$

we find

$$R_2(\nu) = - [\hat{k}\ H_\nu^{(1)'}(\hat{k}a_1)H_\nu^{(1)}(ka_1)-k\ H_\nu^{(1)}(\hat{k}a_1)H_\nu^{(1)'}(ka_1)] \,/\, [\hat{k}H_\nu^{(1)'}(\hat{k}a_1)H_\nu^{(2)}(ka_1)-k\ H_\nu^{(1)}(\hat{k}a_1)H_\nu^{(2)'}(ka_1)] \tag{27}$$

a "reflection" coefficient at the boundary. Similarly, from the boundary condition at $r=a_o$, i.e.,

$$\left.\frac{\partial g}{\partial r}\right|_{r=a_o} = -ik\delta g\Big|_{r=a_o} \tag{28}$$

we find

$$R_1(\nu) = - [H_\nu^{(1)'}(ka_o) + i\delta H_\nu^{(1)}(ka_o)] \,/\, [H_\nu^{(2)'}(ka_o) + i\delta H_\nu^{(2)}(ka_o)] \tag{29}$$

where

$$\delta = \begin{cases} \dfrac{\sqrt{\eta-1}}{\eta}, & \text{vertical polarization} \\[2ex] \sqrt{\eta-1}, & \text{horizontal polarization} \end{cases} \tag{30}$$

and

$$\eta = \varepsilon_r - \frac{i\sigma}{\omega\varepsilon_o} \tag{31}$$

where σ is the ground conductivity in Siemens/m and ε_r is the relative dielectric constant. Knowing $R_1(\nu)$, $R_2(\nu)$ and $\alpha(\nu)$ allows us to solve for $T(\nu)$ in (19), again using the continuity of g at $r=a_1$ and we obtain

$$T(\nu) = - (i/8)[H_\nu^{(1)}(ka_1) + R_2(\nu)H_\nu^{(2)}(ka_1)] \,/\, [R_2(\nu)-R_1(\nu)]H_\nu^{(1)}(\hat{k}a_1). \tag{32}$$

Similarly, the jump condition for $\hat{g}$ gives

$$\hat{\alpha}(\nu)\ \hat{R}_2(\nu) = (i/8)\ . \tag{33}$$

From the boundary condition at $r=a_2$, we find

$$\hat{R}_2(\nu) = -\{\hat{k}\, H_\nu^{(1)\prime}(\hat{k}a_2)[H_\nu^{(1)}(ka_2) + R_1(\nu)H_\nu^{(2)}(ka_2)]$$
$$- k\, H_\nu^{(1)}(\hat{k}a_2)[H_\nu^{(1)\prime}(ka_2) + R_1(\nu)H_\nu^{(2)\prime}(ka_2)]\} \;/ \tag{34}$$
$$\{\hat{k}\, H_\nu^{(2)\prime}(\hat{k}a_2)[H_\nu^{(1)}(ka_2) + R_1(\nu)\, H_\nu^{(2)}(ka_2)]$$
$$- k\, H_\nu^{(2)}(\hat{k}a_2)[H_\nu^{(1)\prime}(ka_2) + R_1(\nu)H_\nu^{(2)\prime}(ka_2)]\}$$

and

$$\hat{T}(\nu) = \hat{\alpha}(\nu)[H_\nu^{(1)}(\hat{k}a_2) + \hat{R}_2(\nu)H_\nu^{(2)}(\hat{k}a_2)] \;/ \tag{35}$$
$$[H_\nu^{(1)}(ka_2) + R_1(\nu)H_\nu^{(2)}(ka_2)].$$

We now turn our concentration to the aperture integration $a_o \le r' < \infty$ in (17). The geometry of the problem is given in Figure 2, where we will confine our analysis to the case where $a_o \le r_s \le a_1$ and $a_o \le r \le a_2$.

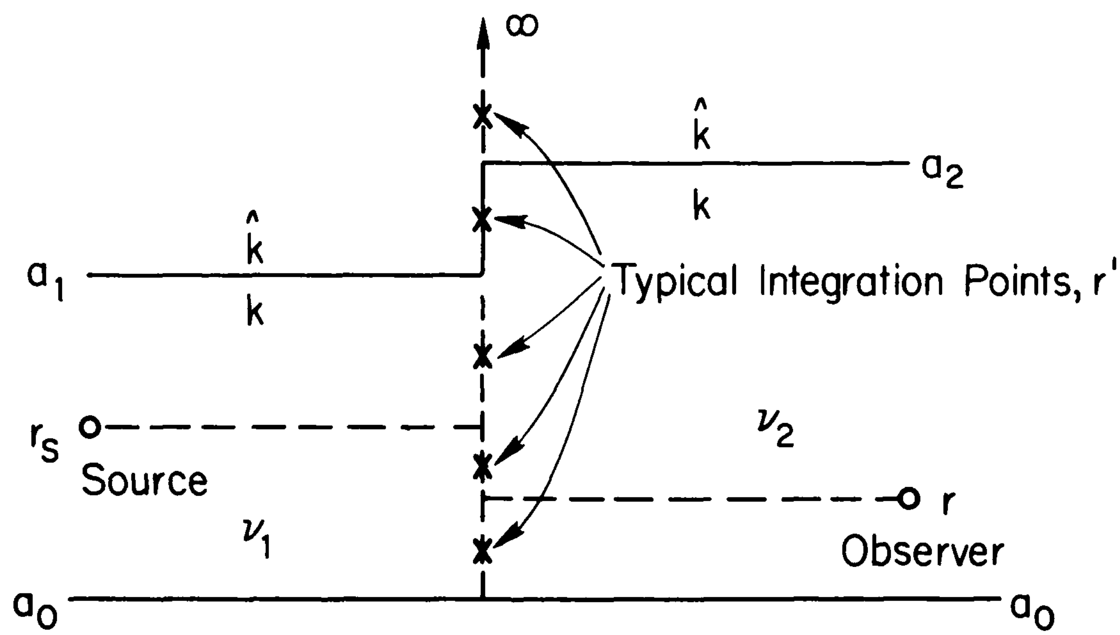

Figure 2. Geometry for aperture integration. Modes in regions (I) and (II) are ν_1 and ν_2, respectively.

Using the "broken" functions in (19) and (20) with appropriate interpretations for the source and observations points (i.e., in the aperture plane the integration point, r', becomes the observation point, r, in Region (I) while the integration point, r', becomes the source point, r_s, for Region II), we have

$$\int_{a_o}^{\infty} \frac{dr'}{r'}\, g_r^{(1)}(r',r_s;\nu_1^2)g_r^{(2)}(r,r';\nu_2^2) = \alpha(\nu_1)\alpha(\nu_2)\psi_{\nu_1}(kr_s)\psi_{\nu_2}(kr) \int_{a_o}^{r}\frac{dr'}{r'}\phi_{\nu_1}(kr')\phi_{\nu_2}(kr')$$

$$+\alpha(\nu_1)\alpha(\nu_2)\psi_{\nu_1}(kr_s)\phi_{\nu_2}(kr) \int_{r}^{r_s} \frac{dr'}{r'}\, \phi_{\nu_1}(kr')\psi_{\nu_2}(kr')+\alpha(\nu_1)\alpha(\nu_2)\phi_{\nu_1}(kr_s)\ \phi_{\nu_2}(kr)\int_{r_s}^{a_1} \frac{dr'}{r'}\ \cdot \tag{36}$$

$$\cdot\ \psi_{\nu_1}(kr')\ \psi_{\nu_2}(kr')$$

$$+\alpha(\nu_2)T(\nu_1)\phi_{\nu_1}(kr_s)\phi_{\nu_2}(kr) \int_{a_1}^{a_2} \frac{dr'}{r'}\, H_{\nu_1}^{(1)}(\hat{k}r')\psi_{\nu_2}(kr')+\hat{T}(\nu_2)T(\nu_1)\phi_{\nu_1}(kr_s)\phi_{\nu_2}(kr) \int_{a_2}^{\infty} \cdot$$

$$\cdot\ \frac{dr'}{r'}\, H_{\nu_1}^{(1)}(\hat{k}r')H_{\nu_2}^{(1)}\ (\hat{k}r')$$

where

$$\begin{aligned} \phi_\nu(kr) &= H_\nu^{(1)}(kr) + R_1(\nu)H_\nu^{(2)}(kr) \\ \psi_\nu(kr) &= H_\nu^{(1)}(kr) + R_2(\nu)H_\nu^{(2)}(kr). \end{aligned} \tag{37}$$

From the differential equation (9) for the radial functions and linear combinations as given in (37) we have

$$\int_{a_o}^{r} \frac{dr'}{r'}\, \phi_{\nu_1}(kr')\phi_{\nu_2}(kr') = \frac{r}{(\nu_1^2-\nu_2^2)}\left[\phi_{\nu_2}(kr)\ \frac{\partial\phi_{\nu_1}}{\partial r'}\bigg|_{r'=r} -\phi_{\nu_1}(kr)\ \frac{\partial\phi_{\nu_2}}{\partial r'}\bigg|_{r'=r}\ \right]$$

$$- \frac{a_o}{(\nu_1^2-\nu_2^2)}\left[\phi_{\nu_2}(ka_o)\ \frac{\partial\phi_{\nu_1}}{\partial r'}\bigg|_{r'=a_o} - \phi_{\nu_1}(ka_o)\ \frac{\partial\phi_{\nu_2}}{\partial r'}\bigg|_{r'=a_o}\ \right] \tag{38}$$

$$= \frac{r}{(\nu_1^2-\nu_2^2)}\left[\phi_{\nu_2}(kr)\ \frac{\partial\phi_{\nu_1}}{\partial r'}\ {}_{r'=r} - \phi_{\nu_1}(kr)\ \frac{\partial\phi_{\nu_2}}{\partial r'}\ {}_{r'=r}\ \right]$$

because $\phi_\nu(kr)$ satisfies (28). Also

$$\int_r^{r_s} \frac{dr'}{r'} \phi_{\nu_1}(kr')\psi_{\nu_2}(kr') = \frac{r_s}{(\nu_1^2-\nu_2^2)} \left[\psi_{\nu_2}(kr_s) \frac{\partial\phi_{\nu_1}}{\partial r'}\bigg|_{r'=r_s} - \phi_{\nu_1}(kr_s) \frac{\partial\psi_{\nu_2}}{\partial r'}\bigg|_{r'=r_s}\right]$$

$$- \frac{r}{(\nu_1^2-\nu_2^2)} \left[\psi_{\nu_2}(kr) \frac{\partial\phi_{\nu_1}}{\partial r'}\bigg|_{r'=r} - \phi_{\nu_1}(kr) \frac{\partial\psi_{\nu_2}}{\partial r'}\bigg|_{r'=r}\right] \tag{39}$$

$$\int_{r_s}^{a_1} \frac{dr'}{r'} \psi_{\nu_1}(kr')\psi_{\nu_2}(kr') = \frac{a_1}{(\nu_1^2-\nu_2^2)} \left[\psi_{\nu_2}(ka_1) \frac{\partial\psi_{\nu_1}}{\partial r'}\bigg|_{r'=a_1} - \psi_{\nu_1}(ka_1) \frac{\partial\psi_{\nu_2}}{\partial r'}\bigg|_{r'=a_1}\right]$$

$$- \frac{r_s}{(\nu_1^2-\nu_2^2)} \left[\psi_{\nu_2}(kr_s) \frac{\partial\psi_{\nu_1}}{\partial r'}\bigg|_{r'=r_s} - \psi_{\nu_1}(kr_s) \frac{\partial\psi_{\nu_2}}{\partial r'}\bigg|_{r'=r_s}\right]. \tag{40}$$

The following integral involves both the wave numbers k and $\hat{k}$; i.e.,

$$\int_{a_1}^{a_2} \frac{dr'}{r'} H_{\nu_1}^{(1)}(\hat{k}r')\psi_{\nu_2}(kr') = \frac{a_2}{(\nu_1^2-\nu_2^2)} \left[\psi_{\nu_2}(ka_2) \frac{\partial H_{\nu_1}^{(1)}(\hat{k}r')}{\partial r'}\bigg|_{r'=a_2} - H_{\nu_1}^{(1)}(\hat{k}a_2) \frac{\partial\psi_{\nu_2}}{\partial r'}\bigg|_{r'=a_2}\right]$$

$$- \frac{a_1}{(\nu_1^2-\nu_2^2)} \left[\psi_{\nu_2}(ka_1) \frac{\partial H_{\nu_1}^{(1)}(\hat{k}r')}{\partial r'}\bigg|_{r'=a_1} - H_{\nu_1}^{(1)}(\hat{k}a_1) \frac{\partial\psi_{\nu_2}}{\partial r'}\bigg|_{r'=a_1}\right] + \frac{2\Delta nk^2}{(\nu_1^2-\nu_2^2)} \cdot \tag{41}$$

$$\cdot \int_{a_1}^{a_2} d'dr' \, H_{\nu_1}^{(1)}(\hat{k}r')\psi_{\nu_2}(kr')$$

and, finally,

$$\int_{a_2}^{\infty} \frac{dr'}{r'} H_{\nu_1}^{(1)}(\hat{k}r')H_{\nu_2}^{(1)}(\hat{k}r') = - \frac{a_2}{(\nu_1^2-\nu_2^2)} \left[H_{\nu_2}^{(1)}(\hat{k}a_2) \left.\frac{\partial H_{\nu_1}^{(1)}(\hat{k}r')}{\partial r'}\right|_{r'=a_2} - H_{\nu_1}^{(1)}(\hat{k}a_2) \left.\frac{\partial H_{\nu_2}^{(1)}(\hat{k}r')}{\partial r'}\right|_{r'=a_2} \right] . \tag{42}$$

Substituting (38), (39), (40), (41) and (42) into (36) gives

$$\int_{a_0}^{\infty} \frac{dr'}{r'} g_r^{(1)}(r',r_s;\nu_1^2)g_r^{(2)}(r,r';\nu_2^2) = \left(\frac{-1}{64} \right) \frac{1}{[R_2(\nu_2)-R_1(\nu_2)][R_2(\nu_1)-R_1(\nu_1)](\nu_1^2-\nu_2^2)} \Bigg\{$$

(1)

$$\psi_{\nu_1}(kr_s)\phi_{\nu_1}(kr)r \left[\phi_{\nu_2}(kr) \left.\frac{\partial\psi_{\nu_2}}{\partial r'}\right|_{r'=r} -\psi_{\nu_2}(kr) \left.\frac{\partial\phi_{\nu_2}}{\partial r'}\right|_{r'=r} \right]$$

(2)

$$+ \phi_{\nu_2}(kr)\psi_{\nu_2}(kr_s)r_s \left[\psi_{\nu_1}(kr_s) \left.\frac{\partial\phi_{\nu_1}}{\partial r'}\right|_{r'=r_s} -\phi_{\nu_1}(kr_s) \left.\frac{\partial\psi_{\nu_1}}{\partial r'}\right|_{r'=r_s} \right]$$

(3)

$$+ \phi_{\nu_2}(kr)\psi_{\nu_2}(ka_1)a_1 \left[\phi_{\nu_1}(kr_s) \left.\frac{\partial\psi_{\nu_1}}{\partial r'}\right|_{r'=a_1} - \psi_{\nu_1}(ka_1) \frac{\phi_{\nu_1}(kr_s)}{H_{\nu_1}^{(1)}(\hat{k}a_1)} \left.\frac{\partial H_{\nu_1}^{(1)}(\hat{k}r')}{\partial r'}\right|_{r'=a_1} \right]$$

(4)

$$+ \phi_{\nu_2}(kr)\, \phi_{\nu_1}(kr_s) \frac{\psi_{\nu_1}(ka_1)}{H_{\nu_1}^{(1)}(\hat{k}a_1)} a_2 \left[\psi_{\nu_2}(ka_2) \left.\frac{\partial H_{\nu_1}^{(1)}(\hat{k}r')}{\partial r'}\right|_{r'=a_2} -H_{\nu_1}^{(1)}(\hat{k}a_2) \left.\frac{\partial\psi_{\nu_2}}{\partial r'}\right|_{r'=a_2} \right]$$

$$(5)$$

$$+\phi_{\nu_2}(kr)\phi_{\nu_1}(kr_s)\,\frac{\psi_{\nu_1}(ka_1)}{H^{(1)}_{\nu_1}(\hat{k}a_1)}\;2\Delta nk^2 \int_{a_1}^{a_2} r'dr'\; H^{(1)}_{\nu_1}(\hat{k}r')\psi_{\nu_2}(kr') \tag{43}$$

$$(6)$$

$$+\phi_{\nu_2}(kr)\phi_{\nu_1}(kr_s)\,\frac{\psi_{\nu_1}(ka_1)}{H^{(1)}_{\nu_1}(\hat{k}a_1)}\;(4i/\pi)\;\frac{[R_2(\nu_2)-R_1(\nu_2)][H^{(1)}_{\nu_1}(\hat{k}a_2)H^{(1)'}_{\nu_1}(\hat{k}a_2)-H^{(1)}_{\nu_1}(\hat{k}a_2)H^{(1)'}_{\nu_2}(\hat{k}a_2)]}{[\hat{k}\;H^{(1)'}_{\nu_2}(\hat{k}a_2)\phi_{\nu_2}(ka_2)-hH^{(1)}_{\nu_2}(\hat{k}a_2)\phi_{\nu'_2}(ka_2)]}\Bigg\}.$$

We will show that terms (1) and (2) in (43) represent the "uncoupled" or zeroth order solution and (6) represents the first order coupling, while the remaining terms; i.e., (3), (4), and (6) are all of second order. Consider the following combination of terms from (3) and (5); i.e.,

$$\phi_{\nu_2}(kr)\phi_{\nu_1}(kr_s)\,\frac{\psi_{\nu_1}(ka_1)}{H^{(1)}_{\nu_1}(\hat{k}a_1)}\,\left[a_2\psi_{\nu_2}(ka_2)\,\frac{\partial H^{(1)}_{\nu_1}(\hat{k}r')}{\partial r'}\bigg|_{r'=a_2} - a_1\psi_{\nu_2}(ka_1)\,\frac{\partial H^{(1)}_{\nu_1}(\hat{k}r')}{\partial r'}\bigg|_{r'=a_1}\right]$$

$$= \hat{k}\phi_{\nu_2}(kr)\phi_{\nu_1}(kr_s)\,\frac{\psi_{\nu_1}(ka_1)}{H^{(1)}_{\nu_1}(\hat{k}a_1)}\,[a_2\psi_{\nu_2}(ka_2)H^{(1)'}_{\nu_1}(\hat{k}a_2) - a_1\psi_{\nu_2}(ka_1)H^{(1)'}_{\nu_1}(\hat{k}a_1)]. \tag{44}$$

We will make use of the following asymptotics

$$H^{(1)}_{\nu}(\nu z) \sim \left(\frac{2}{ka}\right)^{1/3} Wi^{(1)}(t + x)$$

$$H^{(2)}_{\nu}(\nu z) \sim \left(\frac{2}{ka}\right)^{1/3} Wi^{(2)}(t + x) \tag{45}$$

where $Wi^{(1)}$ and $Wi^{(2)}$ are Airy functions with [4]

$$t = \left(\frac{ka}{2}\right)^{2/3}\left[1- \left(\frac{ka}{\nu}\right)^2\right] \quad \text{or} \quad \left(\nu = ka + \left(\frac{ka}{2}\right)^{1/3} t\right) \tag{46}$$

and the dimensionless "radial wavenumber"

$$x = ka/\left(\frac{ka}{2}\right)^{1/3}. \tag{47}$$

Using (45), (46) and (47) we find (44) becomes

$$\hat{k}\left(\frac{2}{kr}\right)^{1/3}\left[Wi^{(1)}(t_2+X_r)+R_1(t_2)Wi^{(2)}(t_2+X_r)\right]\left(\frac{2}{kr_s}\right)^{1/3}\left[Wi^{(1)}(t_1+X_s)+R_2(t_1)Wi^{(2)}(t_1+X_s)\right].$$

$$\cdot\left[a_2\left(\frac{2}{ka_2}\right)^{1/3}\left[Wi^{(1)}(t_2)+R_2(t_2)Wi^{(2)}(t_2)\right]\left(\frac{2}{\hat{k}a_2}\right)^{1/3}Wi^{(1)'}(t_1+x_a+x_D)\right. \tag{48}$$

$$\left.-a_1\left(\frac{2}{ka_1}\right)^{1/3}\left[Wi^{(1)}(t_2+X_a)+R_2(t_2)Wi^{(2)}(t_2+x_a)\right]\left(\frac{2}{\hat{k}a_1}\right)^{1/3}Wi^{(1)'}(t_1+x_D)\right]$$

with

$$x_r = k\,(a_1-r)/\left(\frac{ka_1}{2}\right)^{1/3},$$

$$x_a = k\,(a_2-a_1)/\left(\frac{ka_1}{2}\right)^{1/3},$$

$$x_s = k(a_1-r_s)/\left(\frac{ka_1}{2}\right)^{1/3},$$

$$x_D = 2\Delta n\left(\frac{ka_1}{2}\right)^{2/3}.$$

Now, we see if the radial wavenumber, x_a, is small, (48) becomes

$$\hat{k}(a_2^{1/3}-a_1^{1/3})\left(\frac{2}{kr}\right)^{1/3}\left(\frac{2}{kr_s}\right)^{1/3}\left(\frac{4}{k\hat{k}}\right)^{1/3}\left[Wi^{(1)}(t_2+x_r)+R_1(t_2)Wi^{(2)}(t_2+x_r)\right].$$

$$\cdot\left[Wi^{(1)}(t_1+x_s)+R_2(t_1)Wi^{(2)}(t_1+x_s)\right]\left[Wi^{(1)}(t_2)+R_2(t_2)Wi^{(2)}(t_2)\right]. \tag{49}$$

$$\cdot Wi^{(1)'}(t_1+x_D)$$

which is negligible compared with (1), (2) and (6).

Also from (3) and (5), we find

$$\phi_{\nu_2}(kr)\phi_{\nu_1}(kr_s)a_1 \left[\psi_{\nu_2}(ka_1) \left.\frac{\partial\psi_{\nu_1}}{\partial r'}\right|_{r'=a_1} - \psi_{\nu_1}(ka_1) \left.\frac{\partial\psi_{\nu_2}}{\partial r'}\right|_{r'=a_1}\right]$$

$$\sim 2ika_1\phi_{\nu_2}(kr)\phi_{\nu_1}(kr_s)R_2(t_2)\left(\frac{2}{ka_1}\right)^{2/3}[Wi^{(1)}(t_1)Wi^{(2)}(t_2+x_a) - \tag{50}$$

$$- Wi^{(2)}(t_1)Wi^{(1)}(t_2+x_a)]$$

which is negligible, again compared with (1), (2) and (6). Consider (6), and the rational function

$$\frac{[H^{(1)}_{\nu_2}(\hat{k}a_2)H^{(1)'}_{\nu_1}(\hat{k}a_2) - H^{(1)}_{\nu_1}(\hat{k}a_2)H^{(1)'}_{\nu_2}(\hat{k}a_2)]}{[\hat{k}\,H^{(1)'}_{\nu_2}(\hat{k}a_2)\phi_{\nu_2}(ka_2) - k\,H^{(1)}_{\nu_2}(\hat{k}a_2)\phi'_{\nu_2}(ka_2)]} \sim \tag{51}$$

$$\frac{[Wi^{(1)}(t_2+x_D)Wi^{(1)'}(t_1+x_D+x_a) - Wi^{(1)}(t_1+x_D+x_a)Wi^{(1)'}(t_2+x_D)]}{\hat{k}\,Wi^{(1)'}(t_2+x_D)[Wi^{(1)}(t_2)+R_1(t_2)Wi^{(2)}(t_2)] - k\,Wi^{(1)}(t_2+x_D)[Wi^{(1)'}(t_2)+R_1(t_2)Wi^{(2)'}(t_2)]}$$

which is negligible for small radial wavenumber, x_a. Note, even if $x_D = 0$, the denominator in (52) is non-zero because it involves the Wronskain $Wi^{(1)'}(t_2)Wi^{(2)}(t_2) - Wi^{(1)}(t_2)Wi^{(2)'}(t_2) = -2i/\pi$. Of the terms in (43) remaining, (1) and (2) represent the uncoupled normal modes and are given by

$$\psi_{\nu_1}(kr_s)\phi_{\nu_1}(kr)r\left[\phi_{\nu_2}(kr) \left.\frac{\partial\psi_{\nu_2}}{\partial r'}\right|_{r'=r} - \psi_{\nu_2}(kr) \left.\frac{\partial\phi_{\nu_2}}{\partial r'}\right|_{r'=r}\right]$$

$$= (4i/\pi)[R_1(\nu_2) - R_2(\nu_2)]\,\psi_{\nu_1}(kr_s)\,\phi_{\nu_1}(kr) \tag{52}$$

and

$$\phi_{\nu_2}(kr)\psi_{\nu_2}(kr_s)r_s \left[\psi_{\nu_1}(kr_s) \left.\frac{\partial\phi_{\nu_1}}{\partial r'}\right|_{r'=r_s} - \phi_{\nu_1}(kr_s) \left.\frac{\partial\psi_{\nu_1}}{\partial r'}\right|_{r'=r_s}\right] =$$

$$= (4i/\pi)\ [R_2(\nu_1) - R_1(\nu_1)]\ \phi_{\nu_2}(kr)\psi_{\nu_2}(kr_s)\ , \tag{53}$$

where use of (23) is made. The term involving the coupling between regions (I) and (II) is (5) is given by

$$\phi_{\nu_2}(kr)\phi_{\nu_1}(kr_s)\ \frac{\psi_{\nu_1}(ka_1)}{H^{(1)}_{\nu_1}(\hat{k}a_1)}\ 2\Delta nk^2 \int_{a_1}^{a_2} r'dr'\ H^{(1)}_{\nu_1}(\hat{k}r')\psi_{\nu_2}(kr') \tag{54}$$

Returning to (17), we have the ν_1- and ν_2-integrations remaining. These are easily performed using Cauchy's Theorem; the integrands are analytic except at the simple poles which are solutions of

$$R_1\binom{\nu_1}{\nu_2} = R_2\binom{\nu_1}{\nu_2}. \tag{55}$$

The residues at the simple poles are

$$a_1(t) = (\pi/2i)\left\{\frac{x_D[Wi^{(1)}(t + x_D)]^2}{[Wi^{(2)}(t)Wi^{(1)'}(t+x_D)-Wi^{(2)'}(t)Wi^{(1)}(t+x_D)]^2} - \frac{1}{[Wi^{(2)}(t+x_o)]^2}\right\}^{-1} \tag{56}$$

with

$$x_0 = k(a_{\substack{1\\2}} - a_0)/(ka_{\substack{1\\2}}/2)^{1/3} \tag{57}$$

The ν_1- and ν_2-integrations yield the desired result

$$E(r) = (-1/32)\left\{\frac{1}{(ka/2)^{1/3}} \sum_m a_1(t_1^m)\phi_{t_1^m}(kr)\psi_{t_1^m}(kr_s)\right.$$

$$\cdot\exp[i(\phi-\phi_s)(ka_1 + i\ (ka_1/2)^{1/3}\ t_1^m\ ]$$

$$+\frac{1}{(ka/2)^{1/3}} \sum_m a_1(t_2^m)\phi_{t_2^m}(kr)\psi_{t_2^m}(kr_s) \tag{58}$$

$$\cdot\exp[i(\phi-\phi_s)(ka_2 + i\ (ka_2/2)^{1/3}\ t_2^m)]\ -$$

$$+\frac{\pi^2\Delta n}{4}(ka/2)^{1/3}\sum_m\sum_n a_1(t_1^m)a_1(t_2^n)\ \frac{\phi_{t_2^n}(kr)\phi_{t_1^m}(kr_s)}{[(\nu_1^m)^2-(\nu_2^n)^2]}\ \frac{\psi_{t_1^m}(ka_1)}{H^{(1)}_{\nu_1^m}(\hat{k}a_1)}$$

$$\cdot\exp[ika_1 + (\phi_o-\phi_s)t_1^m(ka_1/2)^{1/3} + ika_2 + (\phi-\phi_o)t_2^n(ka_2/2)^{1/3}]$$

$$\left.\cdot\ k\int_{a_1}^{a_2} dr'\ H^{(1)}_{t_1^m}(\hat{k}r')\psi_{t_2^n}(kr')\right\}$$

with

$$[(\nu_1^m)^2(\nu_2^n)^2] \cong 2ka_1\left\{k(a_1a_2) + \left(\frac{ka_1}{2}\right)^{1/3}\left[t_1^m\left(\frac{a_2}{a_1}\right)^{4/3} t_2^n\right]\right\} \tag{59}$$

In (58), E(r) is the field normalized by the source intensity, this result is used for numerical experimentation and has been further normalized as W = |E(r)| (-1/32). Therefore, W is dimensionless.

SECTION III

REMARKS

1) The single sums over the modes in (58) represent the field, assuming uncoupled normal modes[5],[6]. The assumption proceeds by taking the boundary conditions to be independent of the coordinate in the direction of propagation, but the boundary conditions in the normal direction are the same that would be applied for perfectly stratified media. It is also assumed that eigenfunctions corresponding to a particular normal mode are orthornormal; i.e.,

$$\int_{a_o}^{\infty} \frac{dr'}{r'} \phi_n(kr')\phi_m(kr') = \delta_{mn} \ .$$

Unfortunately, in many theories, it is often very difficult to justify when to neglect the coupling. The solution given in (58) allows a direct determination of the effect of coupling. In (58) if the coupling is negligible, the solution suggests the angular position of the step is unimportant, and one must only remember which medium he is in; e.g., source in medium (I), observer in medium (II).

2) Cho and Wait[7] gave a derivation for the fields in a stepped model for a non-uniform duct which employed the use of a non-Hilbert space inner product; i.e., $< \phi_n, \phi_m >$, instead of the usual definition in terms of a complex-valued function or ordered pairs with inner product $< \phi_n,\phi_m^* >$. The natural metric

$$\{x-y,\ x-y\}^{1/2}$$

is a real nonnegative quantity and represents the physical quantity power. Recalling a metric space is complete if every Cauchy sequence is a convergent sequence, the usual definition of a Hilbert space is an inner product space which is complete with respect to its <u>natural</u> metric. The Cho and Wait result can be explained by the use of "biorthogonal" coordinates[8]. Let $\{\nu_n\}$ be the set of nonzero eigenvalues of the differential operator

$$\mathcal{L} = \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + rk^2$$

and let $\{\phi_n\}$ be the corresponding eigenfunctions. The nonzero eigenvalues of the adjoint (formal) operation $\tilde{\mathcal{L}}$ are given by $\{\nu_m^*\}$ and the corresponding eigenfunctions will be denoted by $\{\psi_m\}$. Now, take for the set $\{\psi_m\}$

$$\{\psi_m\} = \{\phi_1^* \ \ \phi_2^* ,\cdot \cdot \cdot\}$$

Then, indeed, the innner product will satisfy

$$< \phi_n, \phi_m > = \delta_{mn}.$$

In fact, Cho and Wait's result for $< \phi_n, \phi_m >$ equals our result in (56); i.e., $< \phi_n, \phi_m > = a_1(t)$. This yields the interesting conclusion that the Cho-Wait inner product will equal zero if and only if $a_1(t)$ equals zero which requires the existence of a double root! This can occur even in a single section duct. Proof: Since the denominator of $a_1(t)$ is a rational function, the only singularities it can have in the entire complex tplane are poles. A double root suggests degenerate modes in the two regions; i.e., the spatial distribution of sources across the aperture plane $\phi=\phi_o$ has the same wavelength as the normal modes being driven and a resonance occurs. The integral formulation used here in terms of the Green's function approach completely sidesteps the issue of what the normalized "eigenfunctions" should be in the biorthonormal coordinate approach. It may be that since the residue in the Green's function approach equals the inner product in the biorthornormal case and because the $H^{(1)}_{\nu_m}(kr)$ are dense in our Hilbert space, any function can be approximated to within $\varepsilon>o$ by $\sum_m a_m H^{(1)}_{\nu_m}(kr)$. The problem comes in finding out how to express the a_m's. This is probably an example of a problem where the solution can be found by a Green's function method but only a generalization of the notion of eigenfunctions permit a solution in terms of the latter. The other point is that the residues come out naturally in the Green's function method.

3) The double sum in (58) depends upon the location of the vertical step, ϕ_o, and represents the coupling from mode m to mode n. The magnitude of this term depends, on the electrical step size, $k(a_2-a_1)$, as seen in (58) and (59).

SECTION IV

EXAMPLE

The numerical results for the four "height-gain" curves in Figure 3 correspond to the following choice of parameters:

$$a_0 = 6378 \text{ km}$$
$$a_1 = 6379 \text{ km}$$
$$a_2 = 6379000,\ 6379010,\ 6379020,\ 6379030 \text{ m}$$
$$r_s = 6379 \text{ km}$$
$$f = 100 \text{ MHz}$$
$$\Delta n = 25 \text{ N-units}$$
$$a_0(\phi_0-\phi_s) \cong 100 \text{ km}$$
$$a_0(\phi_s-\phi\ ) \cong 100 \text{ km}$$
$$\delta = 0.3 - i4\text{x}10^{-2}\ (\sigma = .001 \text{ Seimens/m},\ \varepsilon_r = 10)$$

From Figure 3 it appears that a step size of about 20m causes significant change in the height-gain pattern. We will refer to this as the "resonant" step-size. This would correspond to a radial wave number from (47) of about 0.463 radians (i.e., about $\pi/8$). The second limiting criteria for our solution in (58) is the number of modes required for convergence of the series. For the example in Figure 3, 10 modes gave two significant figures. The convergence of the series is dominated by the exponential terms in the series for small m and by the asymptotic decay of the residues for large m; i.e.,

$$a_1(t_m) \sim \exp(-\ 4/3\ t_m^{3/2})/4\ t_m^{3/2}$$

where for $t_m < x_0$, the imaginary part of t_m becomes small. In Figure 4, the effect of the number modes is shown for a 10 km separation between source and step and step and observer. At this distance 30 modes are required for convergence.

In Figures 5 and 6, the choice of the parameters is the same as in Figure 3 except the frequency is 300 and 60 MHz respectively. Figure 7 is the same as Figure 4 except the refractive index contrast is 50 N-units. The limiting step size for this case is about 10 m. The resonant step size for 300 MHz is about 5 m for $\Delta n = 25$. In Figures 5, 6 and 7, 10 modes provided adequate convergence of the sums.

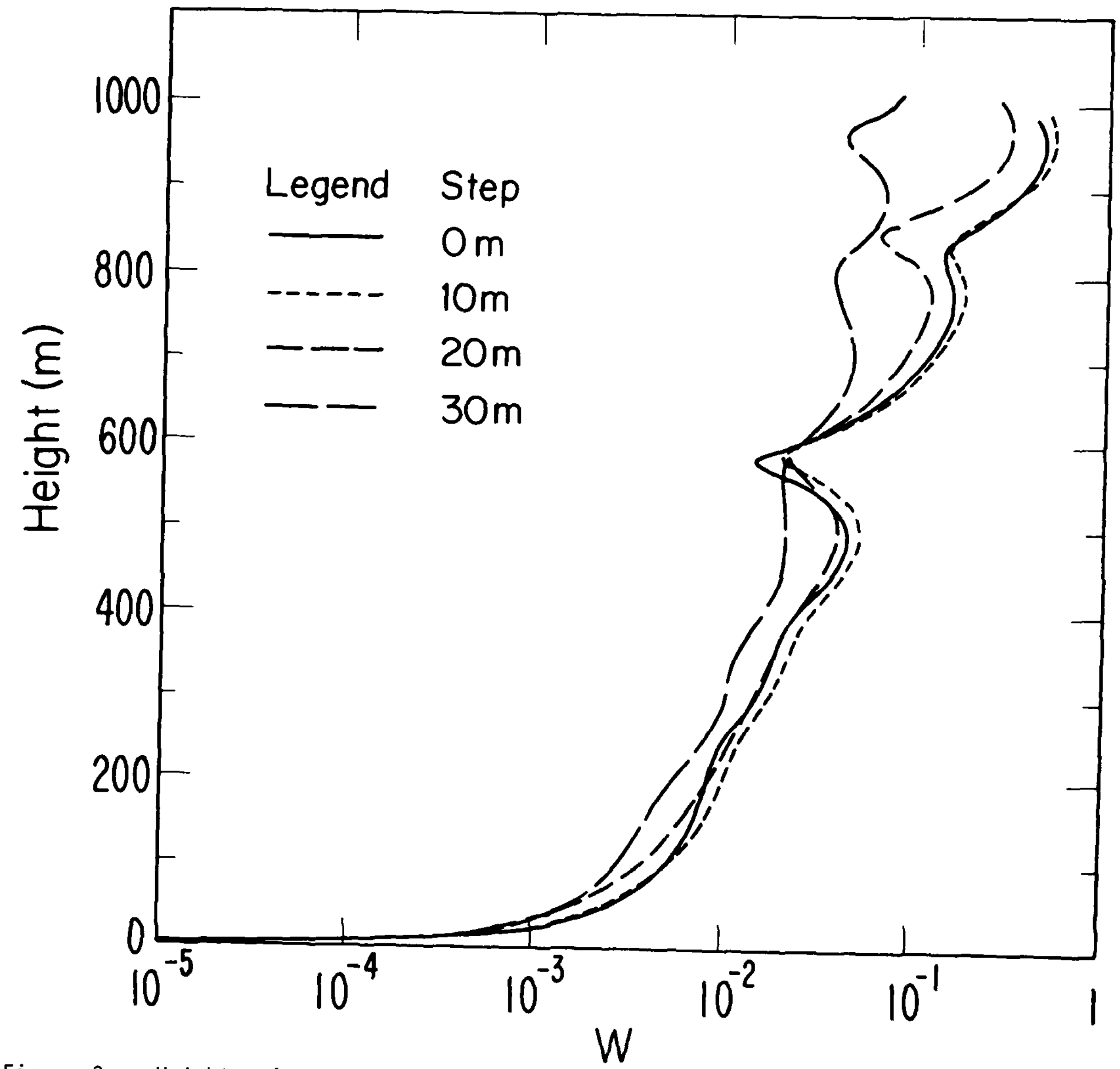

Figure 3. Height-gain curves for a frequency of 100 MHz. Step height shown in legend. Refractive index contrast is 25 N-units. W is the normalized signal strength. The radial wavelength corresponding to the curve "20m" is 0.463 radians.

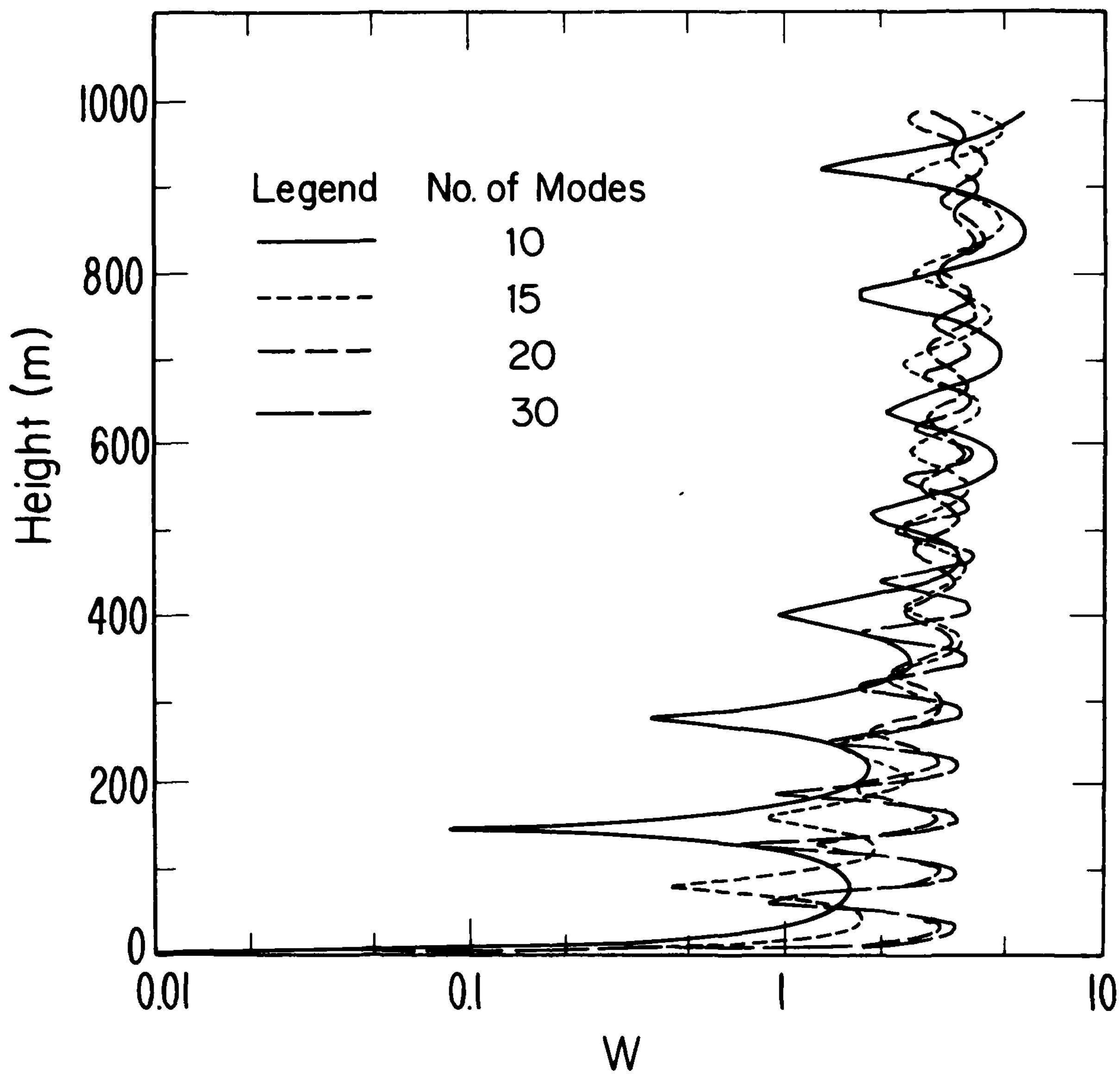

Figure 4. Convergence of solution with increasing number of modes. Distance between source and step and step and observer is 10 km. The frequency is 100 MHz and the refractive index contrast 25 N-units.

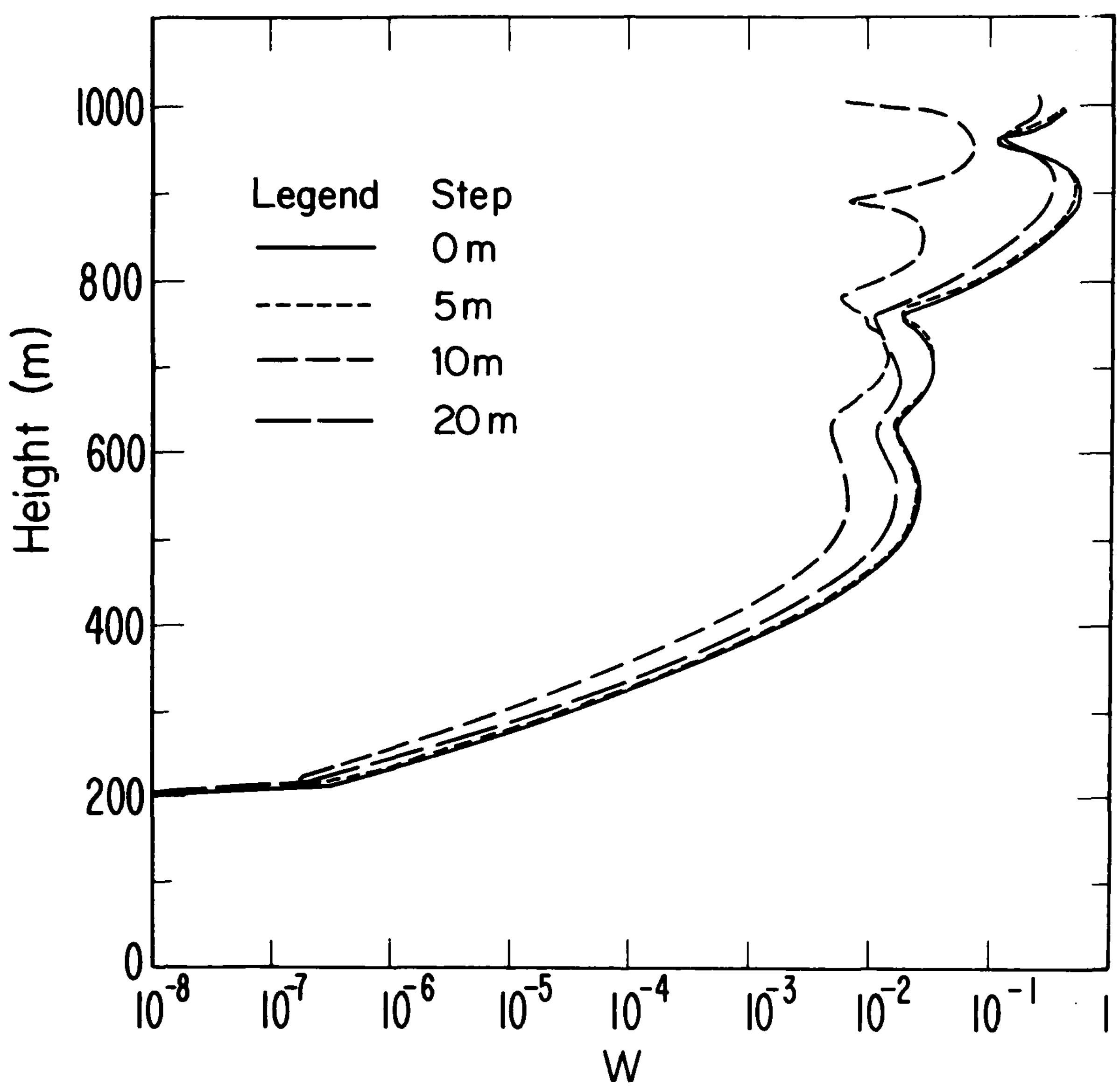

Figure 5. Height-gain run for 300 MHz. W is the normalized signal strength. The refractive index contrast is 25 N-units. 100 km separation between source and aperture plane.

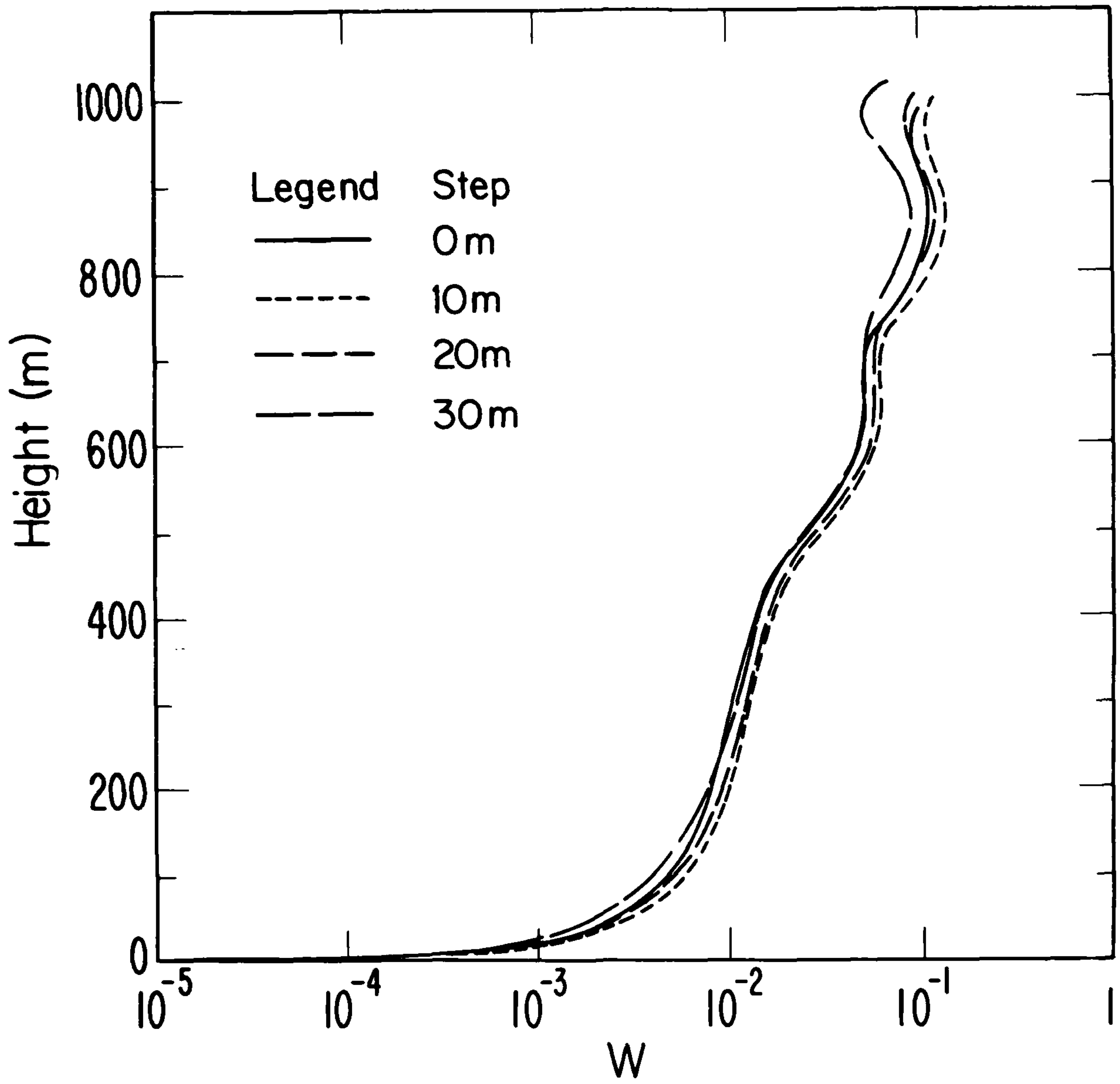

Figure 6. Height-gain run for 60 MHz. W is the normalized signal strength. The refractive index contrast is 25 N-units. 100 m separation.

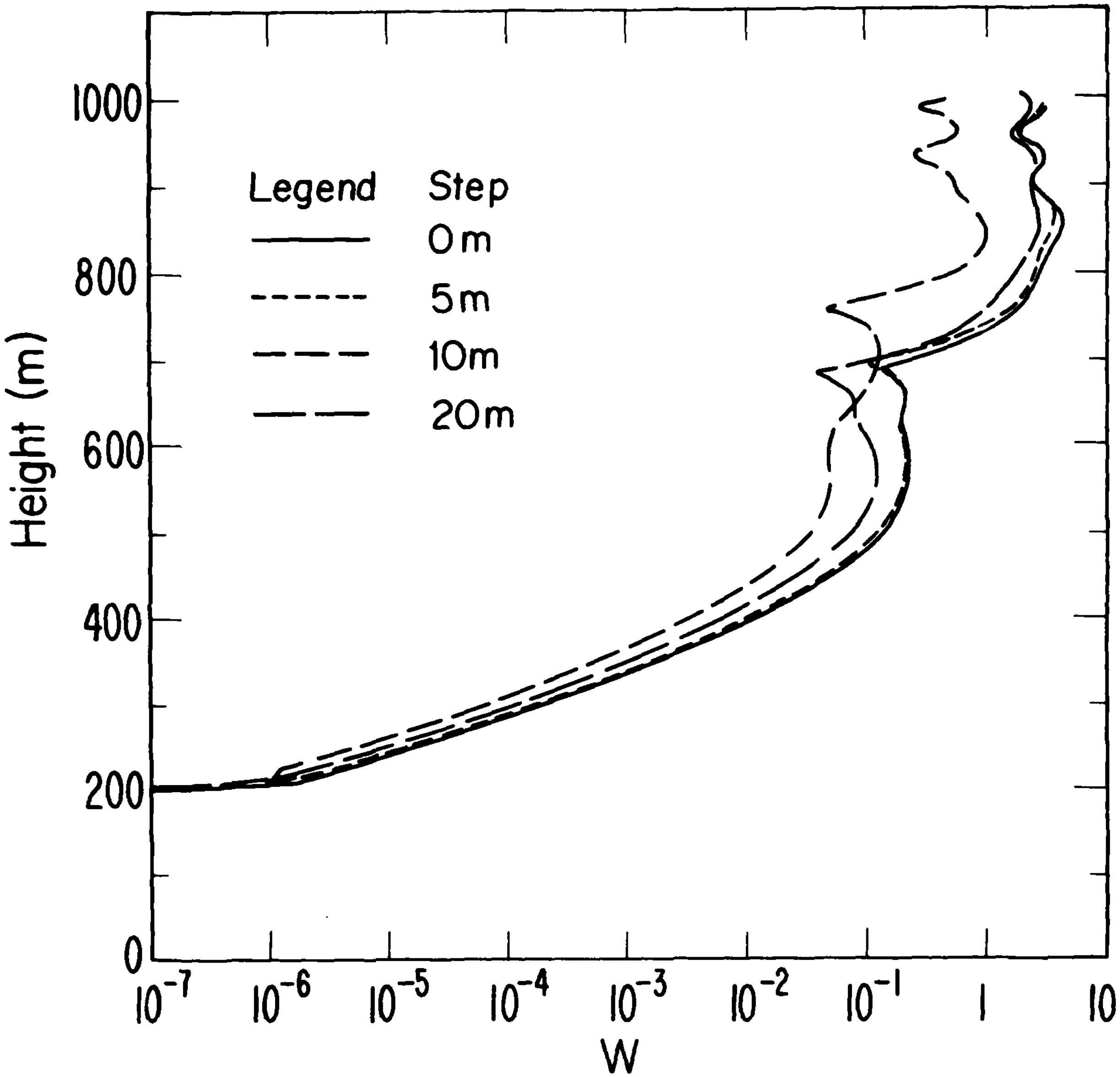

Figure 7. Height-gain run for 300 MHz. W is the normalized signal strength. The refractive index contrast is 50 N-units.

SECTION V

CONCLUSIONS

A Green's function approach is used to examine the effect of varying the step-height in a tropospheric duct with a single step discontinuity. If the electrical height of the step is less than the "radial" separation,

$$k(a_2 - a_1)/(ka/2)^{1/3}$$

the vertical distribution of the field strength agrees with the fields in a duct with no discontinuity. This agrees with a result obtained by Wait and Spies for an ionospheric duct[9]. The location of the step in relation to the source and observer determines the number of modes required for convergence.

SECTION VI

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